Numerical Solution of Partial Differential Equations

Introduction

In mathematics, a **partial differential equation** (**PDE**) is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model.

PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalized similarly in terms of PDEs.

1.1*Partial Differential Equations*

The following equation is an example of a PDE:

$$a\,U_t(x,y,t) + bU_x(x,y,t) + cU_{yy}(x,y,t) = f(x,y,t) \tag{1}$$

where,

- t, x, y are the *independent* variables (often time and space)
- a, b, c and f are known functions of the independent variables,
- U is the *dependent* variable and is an unknown function of the independent variables.
- partial derivatives are denoted by subscript: $\boldsymbol{U_t = \frac{\partial U}{\partial t}, U_x = \frac{\partial U}{\partial x}, U_{yy} = \frac{\partial^2 U}{\partial y^2}}$

The order of a PDE is the order of its highest derivative. A PDE is linear if U and all its partial derivatives occur to the first power only and there are no products involving more than one of these terms. (1) is second order and linear. The dimension of a PDE is the number of independent spatial variables it contains. Equ. (1) is 2D if x and y are spatial variables.

1.2 *Solution of Partial Differential Equation*

Solving PDE means finding the unkown function U. An *analytical* (i.e. exact) solution of a PDE is a function that satisfies the PDE and also satisfies any *boundary* and/ or *initial conditions* given with the PDE. Most PDEs of interest do not have analytical solutions so a *numerical* procedure must be used to find an approximate solution. The approximation is made at discrete values of the independent variables and the approximation scheme is implemented via a computer program. The FDM replaces all partial derivatives and other terms in the PDE by approximations. After some manipulation, a finite difference scheme (FDS) is created from which the approximate solution is obtained.

1.3 *PDE Models*

PDEs describe many of the fundamental natural laws (e.g. conservation of mass) so describe a wide range of physical phenomena. Examples include Laplace's equation for steady state heat conduction, the advection- diffusion equation for pollutant transport, Maxwell's equations for electromagnetic waves, the Navier- stokes equation for fluid flow and many more.

1.4 *Classification of PDEs*

Second order linear PDEs can be formally classified into 3 generic types: elliptic, parabolic and hyperbolic. The simplest examples are:

a) *Elliptic*: e.g. $U_{xx} + U_{yy} = f(x, y)$

This is Poisson's equation or Laplace's equation (when f(x,y) =0) which may be used to model the steady state temperature distribution in a plate or incompressible potential flow. Notice there is no time derivative.

b) *Parabolic*: e.g. $U_t = kU_{xx}$

This is the 1D diffusion equation and can be usedto model the time - dependent temperature distribution along a heated 1D bar.

c) *Hyperabolic*: e.g. $U_{tt} = c^2 U_{xx}$

This is the wave equation and may be used to model a vibrating guitar string or 1D supersonic flow.

1.5 *Types of the boundary of PDEs*

Applications involving ***elliptic equations*** such as a) usually lead to boundary value problems in a region *R*, called a *first boundary value problem* or **Dirichlet problem** if *u* is prescribed on the boundary curve *C* of *R*, a *second boundary value problem* or **Neumann problem** $u_n = \frac{\partial u}{\partial n}$ if (normal derivative of *u*) is prescribed on *C*, and a *third* or **mixed problem** if *u* is prescribed on a part of C and u_n on the remaining part. C usually is a closed curve (or sometimes consists of two or more such curves).

1.6 *Difference Equations for the Laplace and Poisson Equations*

In this section the numeric methods are developed for the two most important elliptic PDEs that appear in applications. The two PDEs are the **Laplace equation**

$$u_{xx} + u_{yy} = 0 \tag{2}$$

and the **Poisson equation**

$$u_{xx} + u_{yy} = f(x, y) \tag{3}$$

The starting point for developing the numeric methods is the idea that the partial derivatives of these PDEs can be replace by corresponding **difference quotients**. To develop this idea, starting with the Taylor formula and obtain:

$$u(x + h, y) = u(x, y) + hu_x(x, y) + \tfrac{1}{2}h^2u_{xx}(x, y) + \tfrac{1}{6}h^3u_{xxx}(x, y) + \cdots \tag{4-a}$$

$$u(x - h, y) = u(x, y) - hu_x(x, y) + \tfrac{1}{2}h^2u_{xx}(x, y) - \tfrac{1}{6}h^3u_{xxx}(x, y) + \cdots \tag{4-b}$$

Subtract (4b) from (4a), neglect terms in h^3, h^4, ..., and solve for u_x. Then

$$u_x(x, y) \approx \frac{1}{2h}[u(x + h, y) - u(x - h, y)]. \tag{5-a}$$

Similarly,

$$u_y(x, y) \approx \frac{1}{2k}[u(x, y + k) - u(x, y - k)]. \tag{5-b}$$

To get second derivatives, adding (4a) and (4b) and neglecting terms in h^4, h^5, ..., and solve for u_{xx}. Then

$$u_{xx}(x, y) \approx \frac{1}{h^2}[u(x + h, y) - 2u(x, y) + u(x - h, y)]. \tag{6-a}$$

Similarly,

$$u_{yy}(x, y) \approx \frac{1}{k^2}[u(x, y + k) - 2u(x, y) + u(x, y - k)]. \tag{6-b}$$

Substitute (6a) and (6b) into the ***Poisson equation*** (3), choosing $k = h$ to obtain a simple formula:

$$u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y-h)-4u(x,y)=h^2 f(x,y). \tag{7}$$

This is a **difference equation** corresponding to (3). Hence for the ***Laplace equation*** (2) the corresponding difference equation is

$$u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y-h)-4u(x,y)=0. \tag{8}$$

h is called the **mesh size**. Equation (8) relates u at (x, y) to u at the four neighboring points shown in Fig. 1. It has a remarkable interpretation: u at (x, y) equals the mean of the values of u at the four neighboring points.

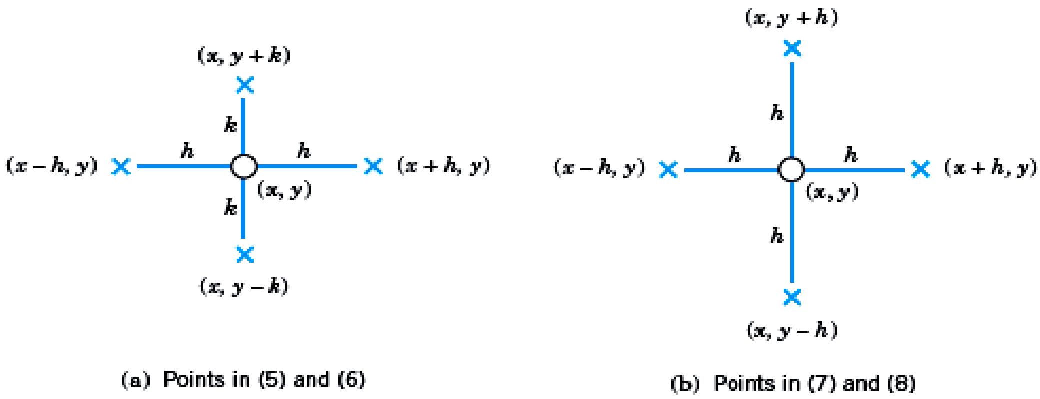


Fig. 1: Points and notations in Eq. 5-8

The approximation of $h^2\nabla^2 u$ in (7) and (8) is a 5-point approximation with the coefficient scheme or **stencil** $\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}$. Now Eq. (7) can be written as:

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u = h^2 f(x,y) \tag{9}$$

1.6.1 ***Dirichlet Problem***

In numerics for the Dirichlet problem in a region R, h is chosen, then introduced a square grid of horizontal and vertical straight lines of distance h. Their intersections are called **mesh points** (or *lattice points* or *nodes*). See Fig. 2.

Then, approximate the given PDE by a difference equation [(8) for the Laplace equation], which relates the unknown values of u at the mesh points in R to each other and to the given boundary values. This gives a linear system of *algebraic* equations. By solving it, get approximations of the unknown values of u at the mesh points in R and note that the number of equations equals the number of unknowns.

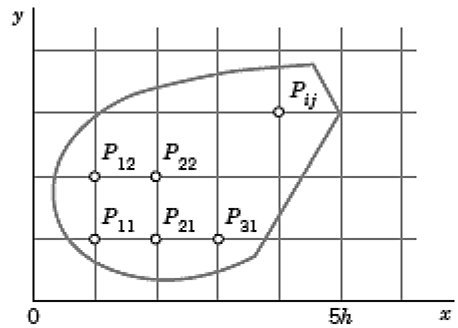


Fig.2: Region in the $x\,y$ – plane is covered by a grid of mesh h, also showing mesh points

$$p_{11} = (h,h), \dots\dots, p_{ij} = (ih, jh)$$

With this notation Eq. (8) can be written for any mesh point p_{ij} in the form:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \qquad (10)$$

<u>Example 1</u>:

The four sides of a square plate of side 12 cm, made of homogeneous material, are kept at constant temperature $0^{\circ}C$ and $100^{\circ}C$as shown in Fig. 3-a. Using a (very wide) grid of mesh 4 cm and find the (steady-state) temperature at the mesh points.

<u>Solution</u>:

The problem is a Dirichlet problem and the grid is chosen as shown in Fig. 3-b

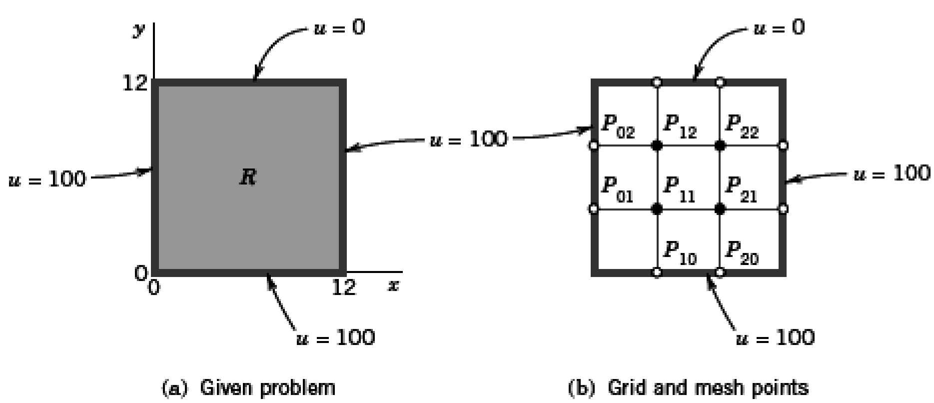


Fig. 3: Example 1

and consider the mesh points in the order $p_{11}, p_{21}, p_{12}, p_{22}$ as used in Eq. (10) and, in each equation, take to the right all the terms resulting from the given boundary values. Then, obtain the system:

$$\begin{aligned} -4u_{11} + u_{21} + u_{12} \qquad\quad &= -200 \\ -u_{11} - 4u_{21} \qquad + u_{22} &= -200 \\ u_{11} \qquad - 4u_{12} + u_{22} &= -100 \\ u_{21} + u_{12} - 4u_{22} &= -100. \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} -200 \\ -200 \\ -100 \\ -100 \end{bmatrix}$$

In practice, one would solve such a small system by the Gauss elimination, finding:

$$u_{11} = u_{21} = 87.5 \text{and} u_{12} = u_{22} = 62.5$$

Example 2:

Solve, $\nabla^2 u = -10x$ for the shown square 3×3 with h =1 as in Fig. 4.

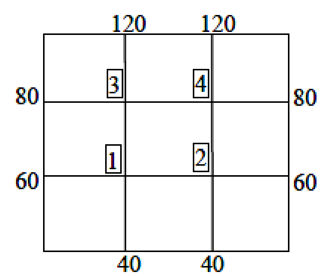


Fig. 4: Example 2

Solution:

60 + 40 + u_2 + u_3 - 4u_1 = (1)^2 (-10*1)

40 + 60 + u_4 + u_1 - 4u_2 = (1)^2 (-10*2)

120 + 80 + u_1 + u_4 - 4u_3 = (1)^2 (-10*1)

80 + 120 + u_3 + u_2 - 4u_4 = (1)^2 (-10*2)

$$\Rightarrow \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -110 \\ -120 \\ -210 \\ -220 \end{bmatrix}$$

Solving the system, get

$$u_1 = 68.75, u_2 = 71.25 , u_3 = 93.75 \text{ and } u_4 = 96.25$$

Example 3:

Solve Poisson's equation $\nabla^2 u = 4y$ on a thin plate of dimension 1.5×2 units,

$u(0,y) = 20y$, $u(1.5,y) = 20y$, $u(x,0) = 0$, $u(x,2) = 30 + 20x(x - 1.5)$, with

$$h = k = 0.5$$

Solution:

The problem is symmetric along the axis A-A as shown in Fig. 5

This means that $u_1 = u_2$,

$u_3 = u_4$ and $u_5 = u_6$

The solution system can be written as:

10 + 0 + u_2 + u_3 - 4u_1 = $(0.5)^2$ (2)

20 + u_1 + u_4 + u_5 - 4u_3 = $(0.5)^2$ (4)

30 + u_3 + u_6 + 20 - 4u_5 = $(0.5)^2$ (6)

$$\begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_5 \end{bmatrix} = \begin{bmatrix} -9.5 \\ -19 \\ -48.5 \end{bmatrix}$$

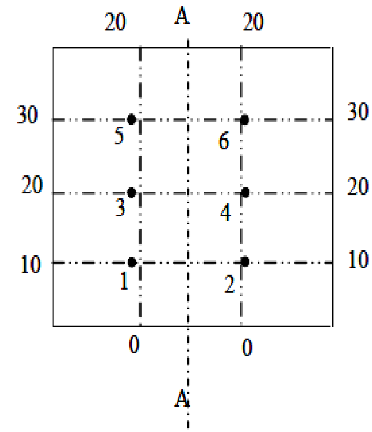


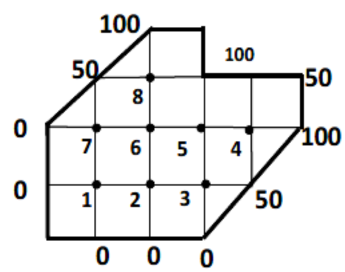
Fig. 5: Example 3

Solving the solution system, get

$u_1 = 8.64$, $u_3 = 16.43$, $u_5 = 21.64$ and from symmetry, $u_2 = 8.64$, $u_4 = 16.43$, $u_6 = 21.64$.

Quiz 1:

Using Matlab code and solve Laplace equation for the internal mesh points with $h = k = 0.5$ and the boundary conditions as shown in Figure.

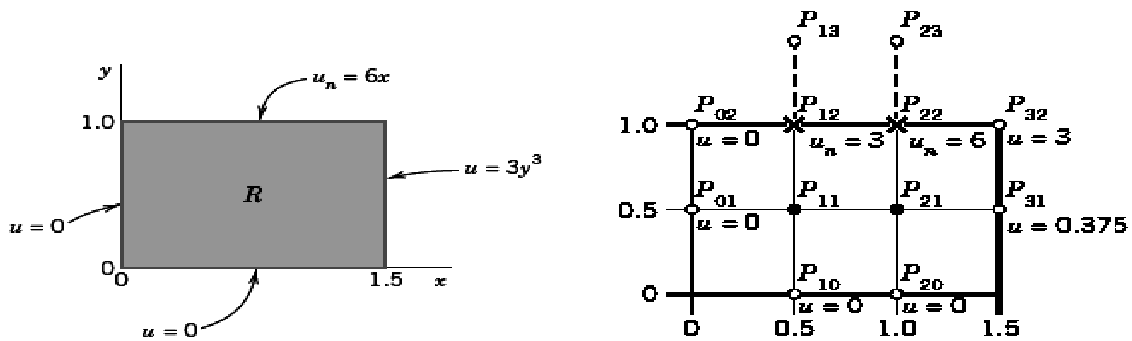


1.6.2 ***Neumann* and *Mixed Problem***

In solving **Neumann** and **mixed problems** a new situation was appeared because there are boundary points at which the (outer) **normal derivative** $u_n = \frac{\partial u}{\partial n}$ of the solution is given, but u itself is unknown since it is not given. To handle such points we need a new idea. This idea is the same for Neumann and mixed problems. Hence we may explain it in connection with one of these two types of problems. We shall do so and consider a typical example as follows.

Example 4:

Solvethe mixed boundary value problem or the Poisson equation$\nabla^2 u = f(x, y) = 12xy$ in the region and for the boundary conditions as shown in Fig. 6-a



(a) Region R and the boundary values (b) Grid (h = 0.5)

Fig. 6: Mixed boundary value problem in Example 4

Solution:

Start the solution by using the grid shown in Fig.6-b where $h = 0.5$ and the right hand side is $h^2 f(x, y) = 0.5^2\ 12xy = 3xy$. From the formulas $u = 3\, y^2$and $u_n = 6x$ given on the boundary we compute the boundary data as:

$$u_{31} = 0.375\ , u_{32} = 3\text{and}\frac{\partial u_{12}}{\partial n} = \frac{\partial u_{12}}{\partial y} = 6 * 0.5 = 3\ , \frac{\partial u_{22}}{\partial n} = \frac{\partial u_{22}}{\partial y} = 6 * 1 = 6$$

The two equations corresponding to $P_{11}$and P_{21}as follows:

- 4u_{11} + u_{21} + u_{12}=0.5^2 $12xy$= 0.25 *12* (0.5* 0.5) – 0 = 0.75

(11)

u_{11} -4 u_{21} + u_{22} =0.5^2 $12xy$ = 0.25 *12* (1* 0.5) – 0.375 = 1.125

The only difficulty with these equations seems to be that they involve the unknown values $u_{12}$and $u_{21}$of u at$P_{12}$and P_{22}on the boundary, where the normal derivative is given, instead of u; but we shall overcome this difficulty as follows.

The idea that help us

- ✓ imagine the region R extended above to the first row of external mesh points (y=1.5) as in Fig. 6 – b and the Poisson equation also holds in the extended region.
- ✓ write down two more equations:

u_{11} -4 u_{12} + u_{22}+ u_{13} = 0.5^2 $12xy$ = 0.25 *12* (0.5* 1) – 0 = 1.5 – 0 = 1.5

(12)

u_{21}+u_{12} -4u_{22}+u_{23}= 0.5^2 $12xy$ = 0.25 *12* (1* 1) – 3 = 3 – 3 = 0

- ✓ the two unknowns u_{13} and u_{23} can be rid of them by applying the central difference formula for $\frac{\partial u}{\partial y}$ as following:

$$3 = \frac{\partial u_{12}}{\partial y} = \frac{u_{13} - u_{11}}{2k} = u_{13} - u_{11} \quad \Rightarrow \quad u_{13} = u_{11} + 3$$

$$6 = \frac{\partial u_{22}}{\partial y} = \frac{u_{23} - u_{21}}{2k} = u_{23} - u_{21} u_{23} = u_{21} + 6$$

- ✓ substituting into (12) and simplify, we obtain:

  $2u_{11}$ -4 u_{12} + u_{22} =1.5 – 3 = -1.5

  $2u_{21}$ + u_{12} -4 u_{22} =3 – 3 – 6 = -6

- ✓ Together (12) with (11) this yields, written in matrix form,

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 2 & 0 & -4 & 1 \\ 0 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0.75 \\ 1.125 \\ -1.5 \\ -6 \end{bmatrix}$$

The solution is obtained by Gauss elimination as follows:

$u_{11} = 0.077$, $u_{21} = 0.191$, $u_{12} = 0.866$, $u_{22} = 1.812$

Note that:

If the normal derivative $u_n = \frac{\partial u}{\partial x}$, so

- ✓ the region R will be extend **horizontally** of the external mesh points and the Poisson equation also holds in the extended region.
- ✓ the new unknowns can be rid of them by applying the central difference formula for $\frac{\partial u}{\partial x}$ as following: $\frac{\partial u_{i,j}}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$

Quiz 2:

Solve the Poisson Equation: $\nabla^2 u = f(x, y) = 2(x^2 + y^2)$

With h = **1.0** in x and y direction and

the boundary condition as shown in the Figure.

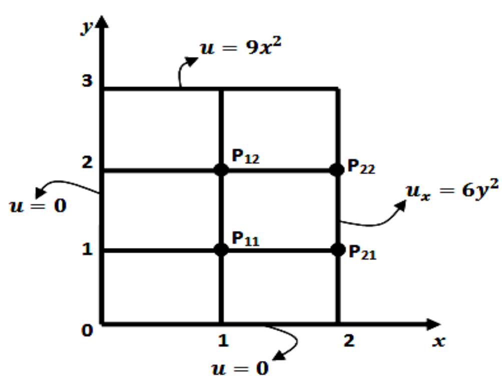


Fig.Quiz 2

1.6.3 ***Irregular Boundary***

If region R in the xy-planehas a simple geometric shape, then we can usually arrange for certainmesh points to lie on the boundary C of R, and we can approximate the partial derivatives. However, if C intersects the grid at points that are not mesh points, then at points close to the boundary we must proceed differently, as follows.

The mesh point O in Fig. 7 is of that kind. For O and its neighbors A and P we obtain from Taylor's theorem

$$\begin{aligned} &\text{(a)} \quad u_A = u_O + ah\frac{\partial u_O}{\partial x} + \frac{1}{2}(ah)^2\frac{\partial^2 u_O}{\partial x^2} + \cdots \\ &\text{(b)} \quad u_P = u_O - h\frac{\partial u_O}{\partial x} + \frac{1}{2}h^2\frac{\partial^2 u_O}{\partial x^2} + \cdots . \end{aligned} \tag{13}$$

We disregard the terms marked by dots and eliminate $\frac{\partial u_0}{\partial x}$. Equation (13 b) times a plus equation (13 a) gives

$$u_A + au_P \approx (1+a)\,u_O + \frac{1}{2}a\,(a+1)\,h^2\frac{\partial^2 u_O}{\partial x^2} .$$

We solve this last equation algebraically for the derivative, obtaining

$$\frac{\partial^2 u_O}{\partial x^2} \approx \frac{2}{h^2}\left[\frac{1}{a(1+a)}u_A + \frac{1}{1+a}u_P - \frac{1}{a}u_O\right].$$

Similarly, by considering the points O, B, and Q,

$$\frac{\partial^2 u_O}{\partial y^2} \approx \frac{2}{h^2}\left[\frac{1}{b(1+b)}u_B + \frac{1}{1+b}u_Q - \frac{1}{b}u_O\right].$$

By addition,

$$\nabla^2 u_O \approx \frac{2}{h^2}\left[\frac{u_A}{a(1+a)} + \frac{u_B}{b(1+b)} + \frac{u_P}{1+a} + \frac{u_Q}{1+b} - \frac{(a+b)u_O}{ab}\right]. \tag{14}$$

Instead of the stencil

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} \text{we now have} \begin{bmatrix} & \frac{1}{b(1+b)} & \\ \frac{1}{1+a} & -\frac{a+b}{ab} & \frac{1}{a(1+a)} \\ & \frac{1}{1+b} & \end{bmatrix}$$

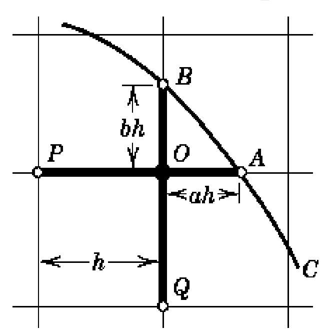


Fig.7: Curved boundary C of a region R,
a mesh point O near C, and neighbors A, B, P, Q

Using the same ideas, you may show that in the case of Fig. 8.

$$\nabla^2 u_O \approx \frac{2}{h^2}\left[\frac{u_A}{a(a+p)}+\frac{u_B}{b(b+q)}+\frac{u_P}{p(p+a)}+\frac{u_Q}{q(q+b)}-\frac{ap+bq}{abpq}u_O\right], \qquad (15)$$

a formula that takes care of all conceivable cases.Instead of the stencil

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} \text{we now have} \quad \left\{\begin{matrix} & \dfrac{1}{b(b+q)} & \\ \dfrac{1}{p(p+a)} & -\dfrac{ap+bq}{abpq} & \dfrac{1}{a(a+p)} \\ & \dfrac{1}{q(q+b)} & \end{matrix}\right\}$$

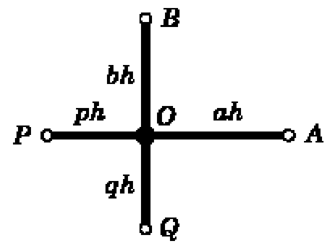


Fig. 8: Neighboring points *A*, *B*, *P*, *Q* of a mesh point *O* and notations in Eq. (15)

> ***Note that:***
> The sum of all five terms must be zero (which is useful for checking).

Example 5:

Find the potential *u* in the region in Fig. 9 that has the boundary values given in that figure; here the curved portion of the boundary is an arc of the circle of radius 10 about (0,0). Use the grid in the figure.

Solution:

For P_{11} and P_{12} we have the usual regular stencil, and for P_{21} and P_{22} we use (15), obtaining

$$P_{11}, P_{12}: \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}$$

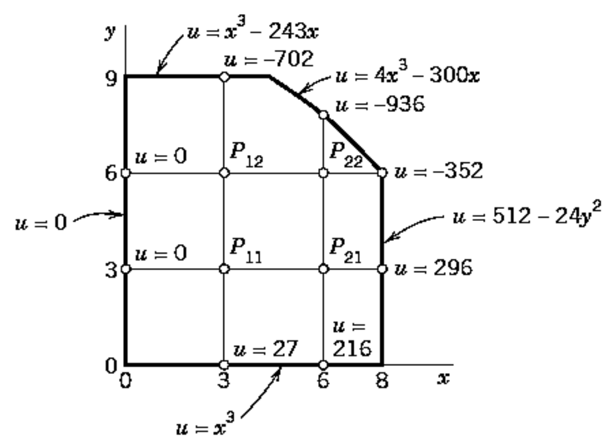


Fig. 9: Region, boundary values of the potential, and grid in Example 5

$$P_{21}: \begin{bmatrix} & 0.5 & \\ 0.6 & -2.5 & 0.9 \\ & 0.5 & \end{bmatrix}, \qquad P_{22}: \begin{bmatrix} & 0.9 & \\ 0.6 & -3 & 0.9 \\ & 0.6 & \end{bmatrix}$$

We use this and the boundary values and take the mesh points in the usual orderP_{11}, P_{21},P_{12} and P_{22}. Then we obtain the system

- 4 u_{11}+ u_{21} +u_{12} = 0 – 27 = - 27

0.6 u_{11}+2.5 u_{21} + 0.5 u_{22} = -0.9*296 – 0.5*216 = -374.4

u_{11} -4u_{12} +u_{22}= 702 + 0 = 702

0.6u_{21}+0.6u_{12}- 3u_{22} = 0.9*352 + 0.9*936 = 1159.2

In matrix form,

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 0.6 & -2.5 & 0 & 0.5 \\ 1 & 0 & -4 & 1 \\ 0 & 0.6 & 0.6 & -3 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} -27 \\ -374.4 \\ 702 \\ 1159.2 \end{bmatrix}$$

Gauss elimination yields the (rounded) values:

$$u_{11} = -55.6 \ , \ u_{21} = 94.2 \ , \ u_{12} = -298.5 \ , \ u_{22} = -436.3$$

<u>Quiz 3:</u>

Use Gauss elimination to solve Laplace equation $\nabla^2 u = 0$, and find the potential u in the indicate grid with boundary values as shown in the Figure.

Where the sloping portion of the boundary is $y = 4.5 - x$

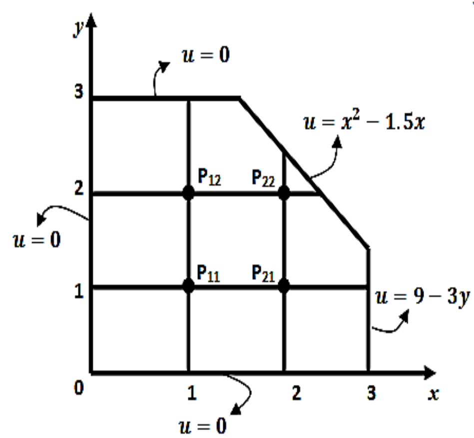


Fig.: Quiz 3

1.7 *Difference Equations for the Heat Equation*

In this section we explain the numeric solution of the prototype of parabolic PDEs, the one-dimensional heat equation.

$$u_{xx} = c^2 u_t \quad (c \text{ constant}).$$

This PDE is usually considered for x in some fixed interval, say, $0 \le x \le L$ and time $t \ge 0$and one prescribes the initial temperature $u(x,0) = f(x)$(f given) and boundary conditions at $x = 0$ and $x = L$ for all $t \ge 0$, for instance, $u(0,t) = f_1$, $u(L,t) = f_2$. Then the **heat equation** and those conditions are:

$$c^2 u_t = u_{xx} 0 \le x \le L\, , t \ge 0 \tag{16}$$

$$u(x,0) = f(x) \qquad \text{(Given Initial displacement)}$$

$$u(0,t) = f_1 \;,\; u(L,t) = f_2 \;\; \text{(Given Boundary conditions)}$$

The *forward* difference formula of the first derivatives is given by

$$u_t = \left.\frac{\partial u}{\partial t}\right|_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k} \tag{17}$$

Substitute Eqs. (17), (6-a) in Eq. (16) to get:

$$\frac{c^2}{k}\left(u_{i,j+1} - u_{i,j}\right) = \frac{1}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \tag{18}$$

$$\left(u_{i,j+1} - u_{i,j}\right) = \frac{k}{c^2 h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

Let $m = \frac{k}{c^2 h^2}$, then

$$u_{i,j+1} = m u_{i+1,j} + (1 - 2m) u_{i,j} + m u_{i-1,j}$$

We can choose k and h to force m to be equal to 0.5, then

$$u_{i,j+1} = \frac{1}{2}\left(u_{i+1,j} + u_{i-1,j}\right) \text{with} m = \frac{k}{c^2 h^2} = 0.5 \tag{19}$$

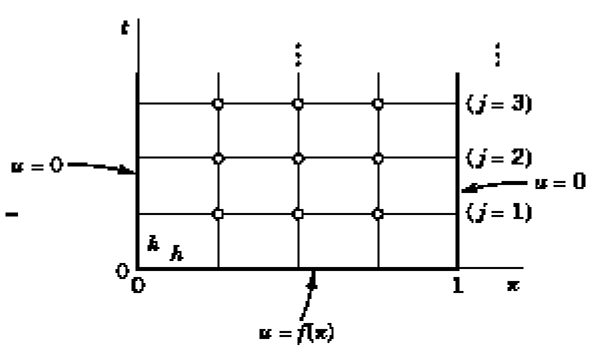


Fig. 10: Grid and mesh points in (18), (19)

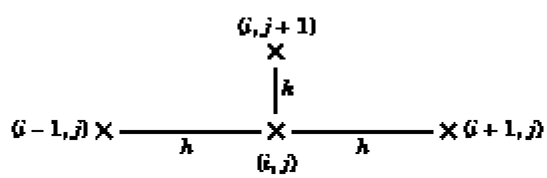


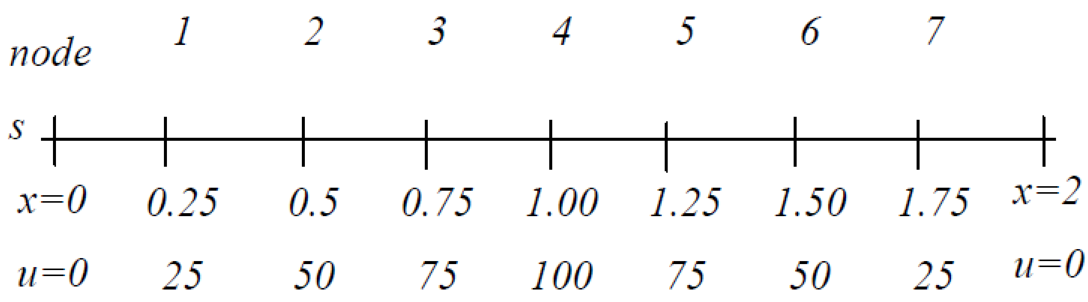
Fig. 11: The four points in (18), (19)

Example 6:

A rod of length 2cm is isolated well. The two ends are kept at $0\,\mathrm{C}^o$ temperature.Find the temperature as a function of x and t. Given $c^2 = 6.6$, $h = 0.25$, $u(x,0) = 100x \quad 0 \le x \le 1$, $u(x,0) = 100(2 - x) \quad 1 \le x \le 2$. Four time steps are required.

Solution:

For $m = 0.5$, we get $k = 0.5 \times 6.6 \times (0.25)^2 = 0.20625$. The following representation of the temperature values of the bar at initial time (at $t = 0$) is shown



For the symmetry of the problem at node 4, we solve for the nodes 1, 2, 3, 4 only.

1-At $t = k = 0.20625$

$$u_1 = \frac{1}{2}(0 + 50) = 25\,, \qquad u_2 = \frac{1}{2}(25 + 75) = 50,$$

$$u_3 = \frac{1}{2}(50 + 100) = 75\,, \qquad u_4 = \frac{1}{2}(75 + 75) = 75$$

2-At $t = 2k = 0.4125$

$$u_1 = \frac{1}{2}(0 + 50) = 25\,, \qquad u_2 = \frac{1}{2}(25 + 75) = 50,$$

$$u_3 = \frac{1}{2}(50 + 75) = 62.5\,, \qquad u_4 = \frac{1}{2}(75 + 75) = 75$$

3-At $t = 3k = 0.61875$

$$u_1 = \frac{1}{2}(0 + 50) = 25\,, \qquad u_2 = \frac{1}{2}(25 + 62.5) = 43.75,$$

$$u_3 = \frac{1}{2}(50 + 75) = 62.5\,, \qquad u_4 = \frac{1}{2}(62.5 + 62.5) = 62.5$$

4-At $t = 4k = 0.825$

$$u_1 = \frac{1}{2}(0 + 43.75) = 21.875\,, \qquad u_2 = \frac{1}{2}(25 + 62.5) = 43.75,$$

$$u_3 = \frac{1}{2}(43.75 + 62.5) = 53.125\,, \qquad u_4 = \frac{1}{2}(62.5 + 62.5) = 62.5$$

X	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2
Nodes		1	2	3	4	5	6	7	
t = 0.825	0	21.875	43.75	53.125	62.5	53.125	43.75	21.875	0
t = 0.61875	0	25	43.75	62.5	62.5	62.5	43.75	25	0
t = 0.4125	0	25	50	62.5	75	62.5	50	25	0
t = 0.20625	0	25	50	75	75	75	50	25	0
t = 0	0	25	50	75	100	75	50	25	0

<u>Quiz 4:</u>

Solve the diffusion equation$u_{xx} = u_t$. For a thin tube 20 cm long with $u(0,t) = 0,\ u(20,t) = 10$and initial condition $u(x,0) = 2$(take $h = 4$cm).Three steps required.

1.8 *Difference Equations for the Wave Equation*

In this section we explain a standard method for the prototype of a hyperbolic PDE, the **wave equation**:

$$\rho^2 u_{xx} = u_{tt} \quad (\rho \text{ constant}).$$

This PDE is usually considered for x in some fixed interval, say, $0 \le x \le L$ and time $t \ge 0$ and one prescribes the initial temperature $u(x,0) = f(x)$(f given) and boundary conditions at $x = 0$ and $x = L$ for all $t \ge 0$, for instance, $u(0,t) = f_1$, $u(L,t) = f_2$. Then the **wave equation** and those conditions are:

$$u_{tt} = \rho^2 u_{xx} 0 \le x \le L , t \ge 0 \tag{20}$$

$$u(x,0) = f(x)(\text{Given Initial displacement})$$

$$u_t(x,0) = g(x)(\text{Given Initial velocity})$$

$$u(0,t) = f_1 \ , \ u(L,t) = f_2 \qquad (\text{Given Boundary conditions})$$

Replacing the derivatives by difference quotients as before, we obtain from (20):

$$\frac{1}{k^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = \frac{\rho^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \tag{21}$$

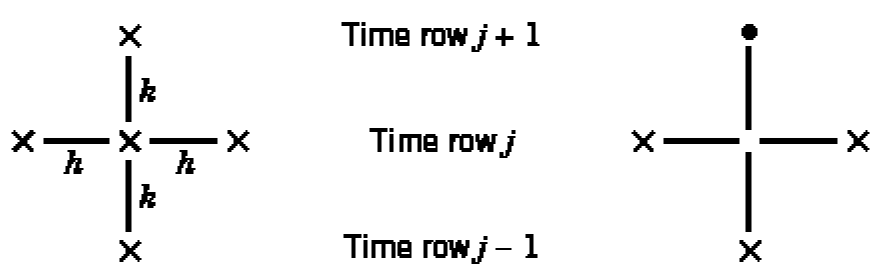
$$\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = \frac{\rho^2 k^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

Let $m^2 = \frac{\rho^2 k^2}{h^2}$, then

$$u_{i,j+1} = m^2\left(u_{i+1,j} + u_{i-1,j}\right) + 2(1 - m^2)u_{i,j} - u_{i,j-1}$$

We can choose k and h to force m to be equal to 1, then

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \text{with} m^2 = \frac{\rho^2 k^2}{h^2} = 1.0 \tag{22}$$



(a) Formula (21) (b) Formula (22)

Fig. 12: Mesh Points used in (21), (22)

Note:

Equation (22) involves 3 time steps $j-1$, $j$and $j+1$. Eq. (22) can be applied directly at every time step ***except at the first time step*** since the two previous steps must be known.

So we ask how we get started and how we can use the initial velocity condition. This can be done as follows.

From$u_t(x,0)=g(x)$ we derive the difference formula

$$u_t|_{i,0}=\frac{1}{2k}\left(u_{i,1}-u_{i,-1}\right)=g_{i,0} \quad \text{, hence} \quad u_{i,-1}=u_{i,1}-2kg_i \tag{23}$$

For $t=0$, that is, $j=0$, Eq. (22) will be

$$u_{i,1}=u_{i+1,0}+u_{i-1,0}-u_{i,-1}$$

Into this we substitute $u_{i,-1}$ as given in (23). We obtain

$$u_{i,1}=u_{i+1,0}+u_{i-1,0}-u_{i,1}+2kg_i$$

and by simplification

$$u_{i,1}=\frac{1}{2}(u_{i-1,0}+u_{i+1,0})+kg_i \tag{24}$$

At the beginning only Equation (24) is applied then, for all the remaining time steps, Equation (22) is applied

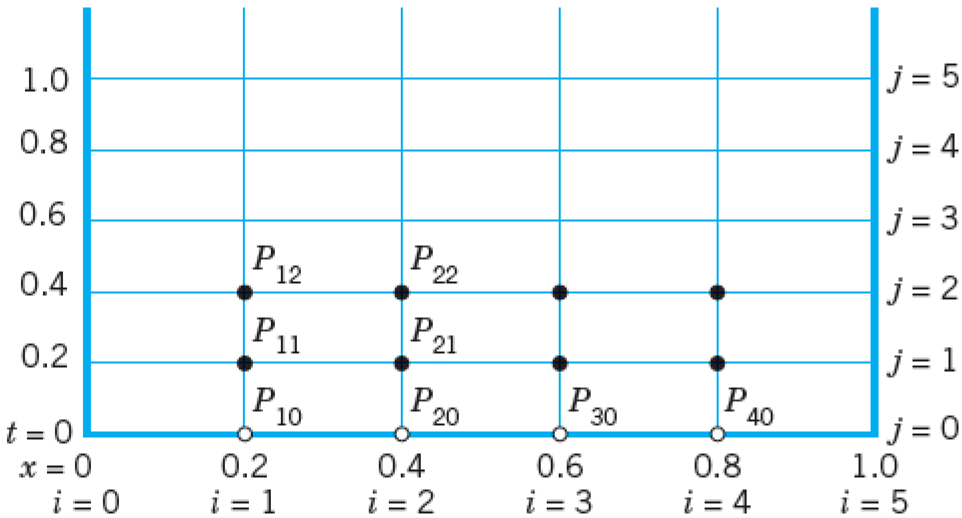
Example 7:

If the governing equation of the string is$u_{xx} = u_{tt}$and the mesh grid as shown in the Figure with $h=0.2$, where

$$u(0,t)=u(1,t)=0, \qquad u(x,0)=sin\pi x, \quad u_t(x,0)=g(x)=0$$

Find the deflection of the string at time t = 1.

Solution:



The initial values $u_{10} = u_{40} = 0.587785$, $u_{20} = u_{30} = 0.951057$

For $j = 0, g(x) = 0$, apply Eq. (24) to get

$$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$$

So,

$$u_{11} = \frac{1}{2}(u_{00} + u_{20}) = \frac{1}{2} \times 0.951057 = 0.475528$$

$$u_{21} = \frac{1}{2}(u_{10} + u_{30}) = \frac{1}{2} \times 0.587785 = 0.769421$$

From symmetry $u_{11} = u_{41}$ and $u_{21} = u_{31}$

For $j = 1$, apply Eq. (22) to get

$$u_{12} = u_{01} + u_{21} - u_{10} = 0.769421 - 0.587785 = 0.181636$$

$$u_{22} = u_{11} + u_{31} - u_{20} = 0.475528 + 0.769421 - 0.951057 = 0.293892$$

From symmetry $u_{32} = u_{22}$ and $u_{42} = u_{12}$; and so on. We thus obtain the following values of the displacement of the string over the first half-cycle:

t	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1$
0.0	0	0.588	0.951	0.951	0.588	0
0.2	0	0.476	0.769	0.769	0.476	0
0.4	0	0.182	0.294	0.294	0.182	0
0.6	0	−0.182	−0.294	−0.294	−0.182	0
0.8	0	−0.476	−0.769	−0.769	−0.476	0
1.0	0	−0.588	−0.951	−0.951	−0.588	0

Quiz 5:

Use the Finite Difference method with ***h* =*k* = 0.2**to approximate the solution of the wave equation

$u_{tt} = u_{xx} \quad if \quad 0 \le x \le 1, t > 0$, where $u(0,t) = u(1,t) = 0 \quad t > 0$ and

$$u(x,0) = \begin{cases} x & if \;\; 0 \le x \le \frac{1}{2} \\ 1 - x & if \;\; \frac{1}{2} < x \le 1 \end{cases}, \; u_t(x,0) = g(x) = sin\pi x \quad 0 \le x \le 1$$

Find the displacement at time ***t* = 0.6** and ***x*** = 0.2, 0.4, 0.6, 0.8.

Exercise

[1] For the square $0 \leq x \leq 4, 0 \leq y \leq 4$ let the boundary temperature be$0°C$ on the horizontal and $50°C$ on the vertical edges. Solve the Laplace Equation to find the temperature at the interior points of a square grid with $h = k = 1$.

[2] For the grid in Fig. 13 compute the potential at the four internal points with the following boundary conditions:

1) u=220 on the upper and lower edges, 110 on the left and right.
2) $u = x^4$ on the lower edge, $u = 81 - 54y^2 + y^4$ on the right,

$u = x^4 - 54x^2 + 81$on the upper edge, $u = y^4$ on the left.

[3] Solve Poisson's equation $\nabla^2 u = -2$on a square of unit length. Each side is divided to 4 equal parts. Find the values of the internal points that

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0$$

[4] Solve the boundary value problem for the Poisson equation$\nabla^2 u = 2(x^2 + y^2)$ in the region shown in Fig. 13and for the boundary conditions
$u(0, y) = u(x, 0) = 0$, $u(3, y) = 2y^2$, $u(x, 3) = 9x^2$.

[5] Solve the Poisson equation in problem 4 for the region and the boundary conditions as shown in Fig. 14

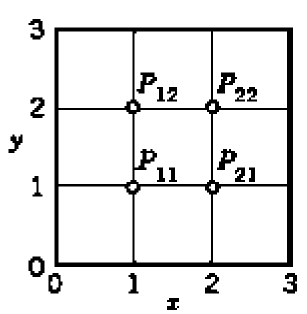


Fig. 13

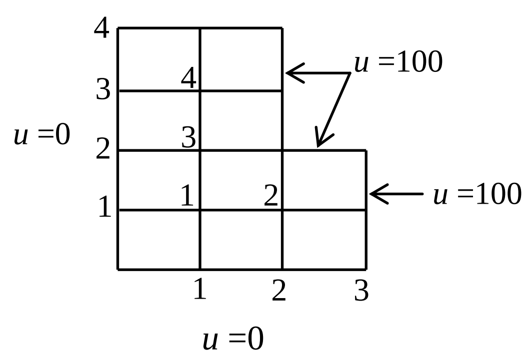


Fig. 14

[6] Solve $\nabla^2 u = -\pi^2 y sin\frac{1}{3}\pi x$ for the grid shown in Fig. 15 and $u_y(1,3) = u_y(2,3) = \frac{1}{2}\sqrt{243}$, $u = 0$ on the other three sides of the square.

[7] Use Gauss elimination to solve Laplace equation $\nabla^2 u = 0$, and find the potential u in the region by using the grid, with h=1 in x and y direction, and the boundary values as shown in the Fig. 16

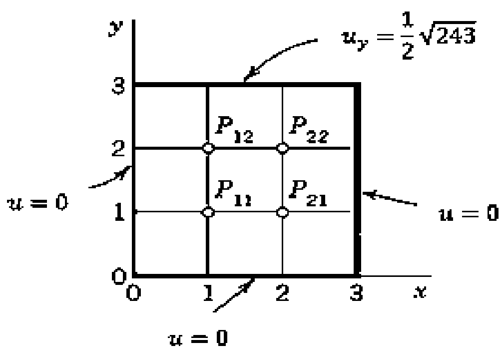


Fig. 15

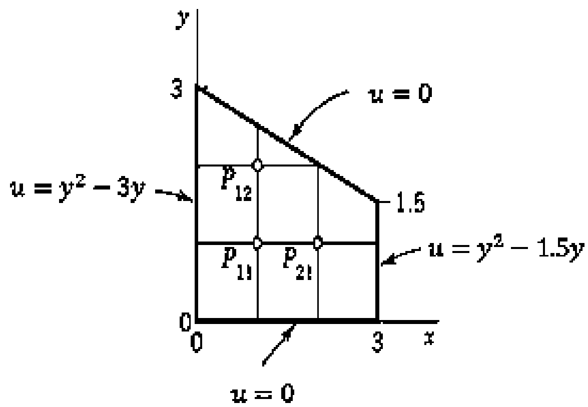


Fig. 16

[8] In a laterally insulated bar of length 1 let the initial temperature be $f(x) = x$ if $0 \le x < 0.5$, $f(x) = 1 - x$ if $0.5 \le x \le 1$ and $u(0,t) = u(1,t) = 0$. Find the temperature $u(x, t)$ in the laterally bar with h =0.2, k = 0 after 5 steps.

[9] Solve the heat equation and let the boundary conditions be $u(0,t) = u(1,t) = 0$ where: $f(x) = x(1 - x)$, h =0.1 (5 steps are required)

[10] Using the present method, solve the wave equation with $h = k = 0.2$ for the given initial deflection $f(x)$ and initial velocity equal $g(x) = 0$ on the given t – interval , $0 \le t \le 1$ and

$f(x) = x$ if $0 \le x < 0.2$,

$f(x) = 0.25(1 - x)$ if $0.2 \le x \le 1$

[11] **Zero initial displacement.** If the string governed by the wave equation $u_{tt} = u_{xx}$ starts from its equilibrium position with initial velocity $g(x) = sin\pi x$, what is its displacement at time t = 0.4 and x = 0.2, 0.4, 0.6, 0.8 with h =k =0.2

Appendix A

Solving a System of Equations Using MATLAB

Left division **** : Left division can be used to solve a system of n equations written in matrix from [a][x]= [b], where [a] is the (n x n) matrix of coefficients, [x] is an (n x 1) column vector of the unknowns, and [b] is an (n x 1) column vector of constants.

x = alb

For example, the solution of the system of equations

$$
\begin{aligned}
4x_1 - 2x_2 - 3x_3 + 6x_4 &= 12 \\
-6x_1 + 7x_2 + 6.5x_3 - 6x_4 &= -6.5 \\
x_1 + 7.5x_2 + 6.25x_3 + 5.5x_4 &= 16 \\
-12x_1 + 22x_2 + 15.5x_3 - x_4 &= 17
\end{aligned}
$$

is calculated by (Command Window):

```
>> A=[4 -2 -3 6; -6 7 6.5 -6; 1 7.5 6.25 5.5; -12 22 15.5 -1];
>> b=[12; -6.5; 16; 17];
>> x=A\b
x =
2.0000
4.0000
-3.0000
0.5000
```